SOME OBSERVATIONS ON THE SIEGEL FORMULA*

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This is a summary of some of our results in a certain area of mathematics that has been created by Siegel (and, among others, by Minkowski). It concerns the theory of quadratic forms and, in particular, an identity which involves on one hand an integral of a theta-series and on the other hand an Eisenstein series. This theory has been reconsidered and, in some aspects, completed by Weil from the adelic viewpoint; he also gave the name "Siegel formula" to the above-mentioned identity. It has turned out that, for a classical group G defined over an algebraic number field k, one has a Siegel formula. Meanwhile, Mars obtained a similar identity for a special k-form of the exceptional simple group of type E_6.

Our first observation was to recognize the possibility that the Siegel formula may depend on which "realization" one takes for G as a matrix group (defined over k). This suggested considering not only the group G but also its rational representation defined over k and thus freeing ourselves from the time-honored doctrine of taking a "semisimple algebra with an involution" as a starting point. Our second observation was to recognize clearly the fact that a quadratic form is the invariant of the corresponding orthogonal group. Putting these observations together, we were able to formulate a conjectural Siegel formula of sufficient generality and to offer a new example of such involving an invariant of arbitrarily high degree. It appears that this area, the exact scope of which is still unclear, might be called the "arithmetic of invariants."

1. Throughout this paper, we shall denote by Ω a universal domain of characteristic 0 and by k an algebraic number field. We shall use the adelic language and denote by the subscript A the "adelization functor" relative to k. If no ambiguity is expected, we shall sometimes drop the subscript A. Also, if K is a finite algebraic extension of k, we shall denote

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by $R_{K/k}$ the functor called the "restriction of the field of definition from K to k." We refer to Weil [7] for the details.

Let G denote a connected, reductive group, X a vector space (of finite dimension), and $\rho\colon G \to \mathrm{Aut}(X)$ a rational representation of G in X. We assume that G, X, and ρ are defined over k. The basic reference on algebraic groups is Borel [1]. Let $\mathscr{S}(X_A)$ denote the Schwartz-Bruhat space of the locally compact, abelian group X_A. Then, denoting by g a typical element of the unimodular group G_A and by $d\mu(g)$ its Haar measure, we shall consider the following integral:

$$I(\Phi) = \int_{G_A/G_k} \left(\sum_{\xi \in X_k} \Phi(\rho(g)\cdot\xi) \right) \cdot d\mu(g),$$

in which Φ is taken from $\mathscr{S}(X_A)$. We shall be interested in such a representation ρ for which $I(\Phi)$ is absolutely convergent, i.e., the integrand is integrable on G_A/G_k in the usual sense for every Φ in $\mathscr{S}(X_A)$. A criterion for this has been formulated by Weil in the language of the reduction theory [8, p. 20]. In the following, we shall recall this criterion.

Fix a maximal k-split torus T of G. If we apply the functor $R_{k/\boldsymbol{Q}}$ to T, we get a torus T' defined over $\boldsymbol{Q}$. Let T'' denote the unique, maximal $\boldsymbol{Q}$-split torus of T'. Then, the connected component Θ of $(T'')_{\boldsymbol{R}}$ can be considered as a subgroup of T_A. We see that Θ is isomorphic to $\boldsymbol{R}^r$, in which r is the k-rank of G, i.e., the dimension of T. We shall denote a typical element of Θ by θ and its Haar measure by $d\theta$. On the other hand, denoting by G_m the universally split torus of dimension 1, we convert the free abelian group $\mathrm{Hom}(T, G_m)$ into a linearly ordered abelian group. Then we define Θ^+ as the set of those θ satisfying $|\alpha(\theta)|_A \leqq 1$ for every positive root α of G relative to T. The absolute value with the subscript A denotes the idele norm. Let λ denote a weight of ρ relative to T and m_λ its multiplicity in ρ. Then, the aforesaid criterion is that the following integral:

$$\int_{\Theta^+} \left(\prod_{\lambda} \sup(1, |\lambda(\theta)|_A^{-m_\lambda}) \right) \left(\prod_{\alpha > 0} |\alpha(\theta)|_A \right) \cdot d\theta$$

be convergent, in which the second product is extended over the set of positive roots α of G relative to T counted with their multiplicities (in the adjoint representation of G).

It is worthwhile to observe that the adjoint representation of G does not satisfy this criterion unless G_A/G_k is compact. In this case, the criterion is satisfied for every ρ. In the general case, we say that ρ is *admissible over* k if it satisfies this criterion. The admissibility over k does not depend on the choice of T nor on the linear order in $\mathrm{Hom}(T, G_m)$. Furthermore, if K is a finite algebraic extension of k and if ρ is admissible over K, it is admissible over k. We say that ρ is *absolutely admissible* if it is admissible over any such K. We observe that, if G splits over K and if ρ is admissible over K, then ρ is absolutely admissible. The first problem is to determine all absolutely admissible representations.

First of all, if $G^* \to G$ is an isogeny defined over k, ρ is admissible over k if and only if the product of ρ and $G^* \to G$ is admissible over k. Secondly, if ρ is absolutely admissible, G is necessarily semisimple. Therefore, for our purpose, we may assume that G is a connected, simply connected, semisimple group defined over k.

Now, any rational representation ρ of G defined over k decomposes uniquely (up to equivalence over k) into a sum of k-irreducible representations. Moreover, if ρ is admissible over k and decomposes over k into a sum of ρ_1 and ρ_2, they are both admissible over k. Therefore, we shall be interested in k-irreducible, absolutely admissible representations. We observe that the trivial representation (mapping every element of G to 1) is absolutely admissible. In the following, we shall mostly disregard this representation. The following lemma is basic:

Lemma. *If $\rho: G \to \mathrm{Aut}(X)$ is a k-irreducible, absolutely admissible representation of a connected, simply connected, semisimple group G defined over k, there exist an absolutely simple factor G_1 of G defined over a finite algebraic extension K of k, a G-invariant subspace X_1 of X defined over K satisfying $X = R_{K/k}(X_1)$, and a K-irreducible representation $\rho_1: G_1 \to \mathrm{Aut}(X_1)$ such that ρ decomposes into the product of the projection $G \to R_{K/k}(G_1)$ and the representation $R_{K/k}(G_1) \to \mathrm{Aut}(X)$.*

Because of this lemma, we may assume that $G = R_{K/k}(G_1)$. Then, after the identification of the adelizations of G, X, etc., over k and the adelizations of G_1, X_1 etc. over K, the integral I for $\rho: G \to \mathrm{Aut}(X)$ relative to k becomes equal to the integral I for $\rho_1: G_1 \to \mathrm{Aut}(X_1)$ relative to K. Consequently, without losing the generality, we may assume that G is a connected, simply connected, absolutely simple group defined over k. In this case, we can determine, up to making them explicit for an exceptional G, all absolutely admissible representations. This will be based on the following classification theorem:

Theorem 1. *Let G_0 denote a connected, simply connected, simple k-split group. Then, the irreducible constituents of an admissible repre-*

sentation of G_0 over k are "fundamental" in the sense of É. Cartan. Furthermore, modulo the "duality" or the "triality," the list of all admissible representations over k is as follows:

type A_n $(n \geqq 1)$	$p\rho_1 + q\rho_n$ $(p+q \leqq n)$, $\rho_1+\rho_2$, $\rho_2+\rho_n$, ρ_2;
type B_n $(n \geqq 2)$	$p\rho_1$ $(p \leqq n-1)$ *in general, and also*
	$\rho_1+\rho_n$, $2\rho_n$, ρ_n $(n=2,3)$,
	$2\rho_1+\rho_4$, $\rho_1+\rho_4$, ρ_4 $(n=4)$, ρ_5 $(n=5)$;
type C_n $(n \geqq 3)$	$p\rho_1$ $(p \leqq n)$, $\rho_1+\rho_2$, ρ_2 *in general, and also*
	ρ_3 $(n=3)$;
type D_n $(n \geqq 4)$	$p\rho_1$ $(p \leqq n-2)$ *in general, and also*
	$\rho_1+\rho_3+\rho_4$, $\rho_1+\rho_3$ $(n=4)$,
	$2\rho_1+\rho_4$, $\rho_1+\rho_4$, ρ_4 $(n=5)$,
	$\rho_1+\rho_5$, ρ_5 $(n=6)$;
type E_6	$\rho_1+\rho_5$, $2\rho_1$, ρ_1;
type E_7	ρ_1 (of degree 56);
type F_4	$2\rho_1$, ρ_1 (of degree 26);
type G_2	$2\rho_1$, ρ_1 (of degree 7).

The numbering of the fundamental representations is the same as in Chevalley [*2*] except for F_4, where it is reversed. If G is a connected, simply connected, semisimple group defined over k, we represent k-simple factors of G as $R_{K/k}(G_1)$, in which G_1 are absolutely simple. We pick any rational representation of G_1 defined over K which is equivalent (over Ω) to the one in the list. This determines a rational representation of $R_{K/k}(G_1)$, hence of G, defined over k. Take the sum of such representations for all k-simple factors of G. In this manner, we get absolutely admissible representations of G defined over k. If we apply this consideration to k-forms of G_0 of type A–D or to a special k-form of G_0 of type E_6, we will get the representations that have appeared in the works of Siegel [*6*], Weil [*8*], and Mars [*3*]. We refer to Serre [*5*] for the concept of k-forms. We observe that, among the remaining admissible representations for G_0, there are essentially two infinite sequences, one for type A and another for type C.

2. In general, let G denote a reductive group, X a vector space, and $\rho: G \to \mathrm{Aut}(X)$ a rational representation of G in X, all defined over k. Then, the coordinate ring $\Omega[X]$ of X over the universal domain Ω is a graded ring, and G operates on $\Omega[X]$ via ρ as a group of degree-preserving automorphisms. It is well known (cf. [9]) that every ideal J of the ring of invariants $\Omega[X]^G$ has the following property:

$$J \cdot \Omega[X] \cap \Omega[X]^G = J.$$

Consequently, the ring $\Omega[X]^G$ is finitely generated over Ω and, if $I(X)$ denotes the affine variety with $\Omega[X]^G$ as its coordinate ring and $f: X \to I(X)$ the corresponding morphism, f is surjective. We observe that $\Omega[X]^G$ is generated over Ω by G-invariant polynomials in $k[X]$. Consequently, we may assume that $I(X)$ and f are defined over k. We observe also that, if $\Omega[X]^G$ is generated over Ω by algebraically independent, G-invariant polynomials, we may even assume that they are homogeneous. In the following definition, we shall assume that G is connected and semisimple:

Definition. *We say that G operates very nicely on X if* (1) *$I(X)$ is an affine space,* (2) *every fiber $f^{-1}(i)$ contains a G-orbit $U(i)$ of dimension equal to* $\dim(X) - \dim(I(X))$ *such that $f^{-1}(i) - U(i)$ is of codimension* $\geqq 2$, (3) *the morphism f is "submersive" at every point of the Zariski open subset*

$$X^1 = \bigcup_{i \in I(X)} U(i)$$

of X, (4) *the stabilizer subgroup of G at every point x of X' is a semidirect product of a connected, semisimple group and a unipotent group.*

Theorem 2. *Let G denote a connected, semisimple group defined over k such that the absolutely simple factors of its universal covering group are of type A–D. Then, for any absolutely admissible representation $\rho: G \to \mathrm{Aut}(X)$ defined over k, G operates very nicely on X.*

The proof of this theorem goes as follows: First of all, if $G^* \to G$ is the universal covering of G and if ρ^* is the product of ρ and $G^* \to G$, the stabilizer subgroup of G at any point x of X is the image by $G^* \to G$ of the stabilizer subgroup of G^* at x. Therefore, we may replace G by G^* and ρ by ρ^*. We observe also that we are concerned with "absolute properties." Therefore, we may assume that G is simply connected and k-split. Then, using the lemma in the previous section, we can reduce this case to the case where G is simple. The rest depends on the case-by-case examination of all admissible representations. This has been carried out for simple groups of type A–D. In this manner, such classical invariants as the "Pfaffian," the "fundamental cubic form in triality," etc., and some new invariants appear in the picture.

Actually, it is expected that the theorem is true for any connected, semisimple group G defined over k. In fact, it should be possible to prove it directly without using the classification. At any rate, using the classification Stephen J. Haris (of our department) is examining the absolutely admissible representations of exceptional simple groups.

3. Going back to the rational representation $\rho: G \to \mathrm{Aut}(X)$ of a connected, semisimple group G in a vector space X, all defined over k, we take a minimal set of G-invariant, homogeneous polynomials, say $f_1(x), f_2(x), \cdots, f_N(x)$, in $k[X]$ satisfying

$$\Omega[X]^G = \Omega[f_1, f_2, \cdots, f_N].$$

We also fix a non trivial character χ of the adele group k_A which takes the value 1 on k. Furthermore, we shall denote by $|dx|_A$ the Haar measure on X_A normalized by the condition that X_A/X_k is of measure 1. Then, for any function Φ in the Schwartz-Bruhat space $\mathscr{S}(X_A)$, we put

$$E'(\Phi) = \sum_{i^* \in k^N} \int_{X_A} \Phi(x) \cdot \chi\left(\sum_{\alpha=1}^{N} f_\alpha(x) i_\alpha^*\right) \cdot |dx|_A.$$

This is called the *Eisenstein-Siegel series* associated with the morphism $f: X \to I(X)$. We observe that, if this series is absolutely and uniformly convergent for every Φ in any compact subset of $\mathscr{S}(X_A)$, the correspondence $\Phi \to E'(\Phi)$ defines a tempered distribution E' on X_A. Furthermore, this distribution is intrinsic in the sense that it does not depend on the choice of $f_1, f_2, \cdots, f_N$.

On the other hand, choose a subset X' of X and denote by $(X')_k$ the intersection of X' and X_k. Then, denoting by $d\mu(g)$ the Haar measure on G_A normalized by the condition that G_A/G_k is of measure 1, we put

$$I'(\Phi) = \int_{G_A/G_k} \left(\sum_{\xi \in (X')_k} \Phi(\rho(g) \cdot \xi)\right) \cdot d\mu(g),$$

in which Φ is in $\mathscr{S}(X_A)$. If ρ is admissible over k, the correspondence $\Phi \to I'(\Phi)$ defines a tempered, positive measure I' on X. Since it is expected that, for any absolutely admissible representation ρ, G operates very nicely on X, we will take as X' the union of all $U(i)$. With this understanding, we shall propose the following conjecture:

A General Siegel Formula. *If $\rho: G \to \mathrm{Aut}(X)$ is an absolutely admissible representation of a connected, simply connected, semisimple group G in a vector spade X, all defined over k, I' and E' are both defined and they are equal.*

Actually, the assumption that G is simply connected may not be necessary. The above conjectural formula is, so to speak, the main part of a complete Siegel formula, and it consists of $I' = E'$, its analogues for subrepresentations of ρ, and of identities of lesser significance, if not trivial. As for its proof, we can reduce the general case to the case where G is absolutely simple. Therefore, it is enough to verify the conjecture case-by-case using the classification. At any rate, if this is true, we will get a rather natural domain of validity for all Siegel formulas.

On the other hand, in the case when there is only one G-invariant $f(x)$, we can consider the zeta-function of the morphism f. We recall that such a zeta-function, where $f(x)$ is a quadratic form, has been investigated in detail by Siegel [6]. Also, Ono [4] has examined a similarly defined zeta-function in the case where $f(x)$ is an arbitrary polynomial. Now, suppose that $f^{-1}(i)$ is a G-orbit for every $i \neq 0$. Let dx denote the translation-invariant gauge-form on X defined over k and put

$$\theta(x) = f(x)^{-\kappa} \cdot dx,$$

in which $\kappa \cdot \deg(f) = \dim(X)$. Let $\lambda = (\lambda_v)$ denote the standard "convergence factor" of $Y = X - f^{-1}(0)$, i.e., $\lambda_v = (1-(q_v)^{-1})^{-1}$ with q_v denoting the number of elements in the residue field of k for each non archimedian valuation v on k. Then, we get the so-called Tamagawa measure $|\lambda\theta(x)|_A$ on Y_A. Let Φ denote an arbitrary function in $\mathscr{S}(X_A)$ and s a complex variable. Then, the integral

$$Z(s, \Phi) = \int_{Y_A} \Phi(x) \cdot |f(x)|_A^s \cdot |\lambda\theta(x)|_A$$

defines a holomorphic function in $\mathrm{Re}(s) > \kappa$. It is expected that this function has an analytic continuation to the entire s-plane as a meromorphic function with eventual poles of order 1, the number of which is twice the number of G-orbits in $f^{-1}(0)$, and satisfies the functional equation

$$Z(s, \Phi) = Z(\kappa - s, \Phi^*),$$

in which Φ^* is the Fourier transform of Φ on X_A with respect to the normalized measure $|dx|_A$.

4. Before we explain the example that we mentioned in the beginning, we shall point out one of the difficulties that we have encountered in proving the Siegel formula. For the sake of simplicity, assume that there is only one G-invariant $f(x)$. Let k_v denote the completion of k with respect to an arbitrary valuation v on k and χ_v the v-component of the global character χ. Let $|dx|_v$ denote a Haar measure on $X_v = X_k \otimes_k k_v$ and Φ_v a function in the Schwartz-Bruhat space $\mathscr{S}(X_v)$. Consider the function F_v^* on k_v defined by

$$F_v^*(i^*) = \int_{X_v} \Phi_v(x) \cdot \chi_v(f(x)i^*) \cdot |dx|_v .$$

The problem is, then, to find good "asymptotic estimates" of $F_v^*(i^*)$ for all v and the "asymptotic expansions to 2 terms" of $F_v^*(i^*)$ for almost all v, both as i^* tends to ∞ in k_v. It is important that Φ_v is not restricted to any subspace of $\mathscr{S}(X_v)$. As we know, this becomes a problem only when the degree of $f(x)$ is at least 3. It appears that a solution of this problem is the key to settling various conjectures that we have enunciated.

As we have remarked at the end of Section 1, there are essentially two infinite sequences of new absolutely admissible representations. One of them consists of the second fundamental representations for k-forms G of SL_m for $m = 2, 3, \cdots$. For the sake of simplicity, assume that $G = SL_m$ over k and, disregarding the trivial case of an odd m, assume that $m = 2n$. In this case, let X denote the vector space of alternating matrices of degree $2n$ and put $\rho(g) \cdot x = gx^tg$ for g in G and x in X. Then, the Pfaffian $Pf(x)$ of x gives the morphism $Pf: X \to I(X) = \Omega$. Furthermore, we have $U(i) = (Pf)^{-1}(i)$ for every $i \neq 0$. Let U_r denote the set of all x in X satisfying $\mathrm{rank}(x) = 2r$ for $0 \leqq r \leqq n$. Then U_r is a G-orbit for $r \neq n$ and $U_{n-1} = U(0)$. In this case, we have a complete Siegel formula. Let $\omega_r(x)$ denote a gauge-form defined over k on the homogeneous space U_r and $|\omega_r(x)|_A$ the corresponding measure on $(U_r)_A$ for $r \neq n$. Then, we have

$$\int_{G_A/G_k} \Big(\sum_{\xi \in X_k} \Phi(g\xi^tg) \Big) \cdot d\mu(g) = \sum_{i^* \in k} \int_{X_A} \Phi(x) \cdot \chi(Pf(x)i^*) \cdot |dx|_A + \sum_{r=0}^{n-2} \int_{(U_r)_A} \Phi(x) \cdot |\omega_r(x)|_A$$

for every Φ in $\mathscr{S}(X_A)$. It might be amusing to observe that, in the special case when $n = 1$, if we identify x with its (1,2)-coefficient, this becomes the Poisson formula for the adele group k_A (or, the "Thetaformel" in non adelic form).

As for the zeta-function $Z(s, \Phi)$ of the morphism Pf, it has all the properties that we have mentioned. Furthermore, if Φ is of the form $\Phi_0 \otimes \Phi_\infty$ with Φ_0, Φ_∞ in the Schwartz-Bruhat spaces $\mathscr{S}(X_0), \mathscr{S}(X_\infty)$ for the obvious decomposition $X_A = X_0 \times X_\infty$, we have

$$Z(s, \Phi) = \textit{elementary factor} \cdot \prod_{i=0}^{n-1} \zeta(s-2i) \cdot$$

$$\int_{X_\infty} \Phi_\infty(x) \cdot |Pf(x)|_\infty^{s-(2n-1)} \cdot |dx|_\infty ,$$

in which $\zeta(s)$ is the Dedekind zeta-function of k and $|dx|_\infty$, for instance, denotes the product of $|dx|_v$ for all archimedean v's.

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